Algorithms and Complexity

- IA Algorithm I,II
- IB Complexity, Computation

Language and Automata

$L \subseteq \Sigma^{\star}$, $\forall x \in L$, the length $n = |x|$

Reference: Formal Language and Automata

Reduction

Reduction of $L_1 \to L_2$ is a computable function $f : \Sigma_1^{\star} \to \Sigma_2^{\star}$ s.t.

- $\forall x \in \Sigma_1^{\star}. f(x) \in L_2 \Leftrightarrow x \in L_1$.
- every string in L_1 is only mapped by f to a string in L_2.
  - $f(x) \notin L_2 \Leftrightarrow x \notin L_1$.

Polynomial time reducible $L_1 \leq_P L_2$.

- the string $f(x)$ produced by the reduction f on input x
  - must be bounded in length by $p(n)$.

g is polynomial function $L_2 \to L_3$

- transitive as $g \circ f$ is polynomial reduction $L_1 \to L_3$
- closed under composition

Usage,

- L_2 is decidable $\to L_1$ is decidable
  - by polynomial $f(x) \in / \notin L_2$
- L_1 is not decidable $\to L_2$ is not decidable
  - L_1: Halting problem

Complexity Class

Time Complexity

measures computation steps

P class

Polynomial p is of form n^k, k is a constant $O(1)$.

$L \in \mathcal{P} = \bigcup_{k=1}^{\infty} TIME(n^k) \Longleftrightarrow$

For all inputs x, M (deterministic Turing machine)

- M runs within polynomial $p(|x|)$ time
- $\forall x \in L$, M outputs 1, otherwise 0.

NP class

$L \in \mathcal{NP} = \bigcup_{k=1}^{\infty} NTIME(n^k) \iff$

For all inputs $x \in L$,

i. Prover M (non-deterministic TM): $x \rightarrow certificate\ c$, where $|c| < p(|x|)$

- solvable (an accepting computation) by prover M within polynomial $p(|x|)$ time

ii. Verifier V (deterministic TM)

- $\exists c. |c| < p(|x|), (x, c)$ accepted by V running within polynomial $p(|x|)$ time
- polynomially satisfiable / certificate of membership

Complement

$L \in \text{co-}\mathcal{NP} \iff \bar{L} = \Sigma^{\star} \backslash L \in \mathcal{NP}$

For all inputs $x \in L$,

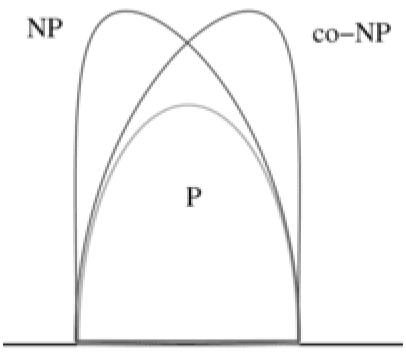
Verifier V (deterministic TM)

- $\exists c. (x, c)$ not accepted by V in polynomial $p(|x|)$ time
- polynomially falsifiable / certificates of disqualification

Relationship

Unknown $\mathcal{NP} \stackrel{?}{=} \mathcal{P}$

- Intersection $P \subseteq NP \cap co\text{-}NP$



Cook-Levin theorem

$L \in \mathcal{NP}\text{-hard}$

- if $\forall A \in \mathcal{NP}, A \leq_P L$.

$L \in \mathcal{NP}\text{-complete}$

- if $L \in \mathcal{NP}$ and $\mathcal{NP}\text{-hard}.$

Space Complexity

measures size of work tape

P class

$PSPACE = \bigcup_{k=1}^{\infty} SPACE(n^k)$

- languages decidable by a deterministic TM with polynomial workspace.

NP class

$NSPACE = \bigcup_{k=1}^{\infty} NSPACE(n^k)$

- languages decidable by a non-deterministic TM with polynomial workspace.

$NL = NSPACE(\log n)$

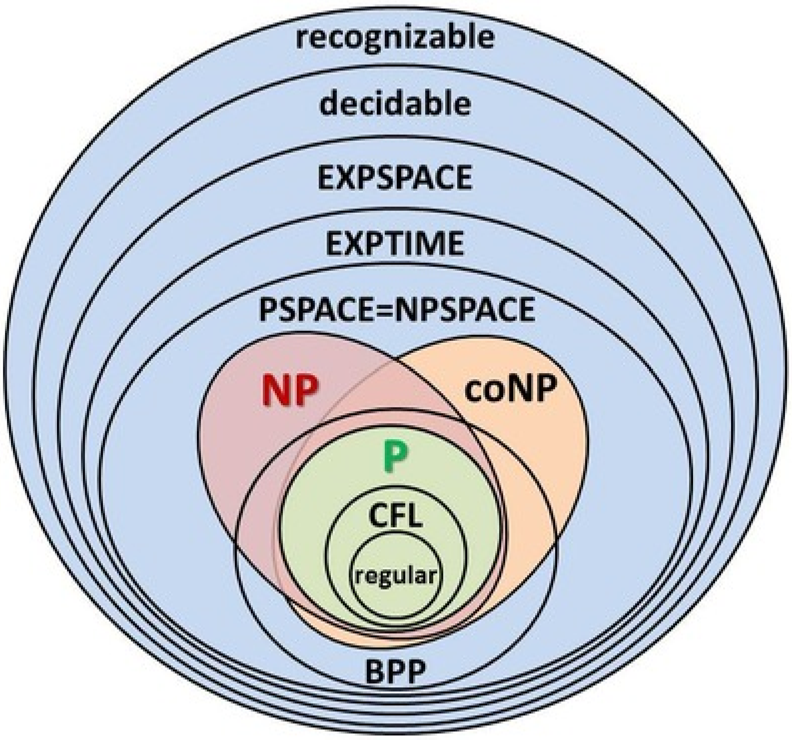
- languages recognisable by a non-deterministic TM with logarithmic workspace.

$L \subseteq NL \subseteq \mathcal{P} \subseteq \mathcal{NP} \subseteq PSPACE \subseteq NPSPACE \subseteq EXP$

- $L, P, PSPACE$ are all closed under complementation
- Unknown $NL \stackrel{?}{=} L$
- Graph Reachability $TIME = O(n^2)$

$$NSPACE(f(n)) \subseteq TIME(k^{\log n + f(n)})$$

- Backtracking $\mathcal{NP}$ with $PSPACE$
- Savitch's Theorem
  - Graph Reachability $SPACE = O((\log n)^2)$
  - When $f(n) = \Omega(\log n)$, $NSPACE(f(n)) \subseteq SPACE(f(n)^2), PSPACE = NPSPACE$.



Hierarchy Theorem

Time

For any constructible function f with $f(n) \geq n$, $TIME(f(n)) \subset TIME(f(2n+1)^2)$ properly contain/subset.

Space

For any pair of constructible functions f and g, with $f = O(g)$ and $g \neq O(f)$, there is a language in $SPACE(g(n))$ that is not in $SPACE(f(n))$.

Lists of Algorithms

- IA Algorithm I,II
- IB Complexity
- II Randomized Algorithm, BioInfo

Decision problem: output starts with ?.

- Negation of decision problem swaps the accept/reject state in TM
  - $SAT = rej, S\bar{A}T = acc$

Optimization problem: output starts with max / min.

Number Theory

$input \in \mathbb{N}$, $n := \#(bits \mathbb{B})$

Algorithm	Input	Output	Complexity	Note
Euclid's algo	(x, y)	$?x = 1$	$O(\log x + \log y)$	in #bits
Prime/COMPosite	$1\{0,1\}^\star$	Prime or Factor	$O(\sqrt{x})$	in #bits
Knapsack	$I = (v_i, w_i), W_M, V_m$	$?\exists I' \subseteq I. W \leq W_M$ $\wedge V \geq V_m$	$\mathcal{NP}$-complete	$X3C <_p Knapsack$
Schedule	/	/	$\mathcal{NP}$-complete	$Knapsack <_p Schedule$
Integer LP	$\sum_i a_i x_i \leq b$	$?x_i \in \{0, 1\}$	$\mathcal{NP}$-complete	CNF-SAT$<_p$

Boolean / nCNF

Variables $X = \{x_1, x_2, ...\}$

Expression $\phi : X$,

- CNF $\phi \equiv C_1 \wedge ... \wedge C_m$
- 3CNF ϕ'
  - each clause C_i is ≤ 3 literals disjunction
  - ϕ conversion to ϕ' in P.

Assignment $T : X \rightarrow \mathbb{B}$

- CVP, $l : X \rightarrow \mathbb{B} \cup \{\wedge, \vee, \neg\}$

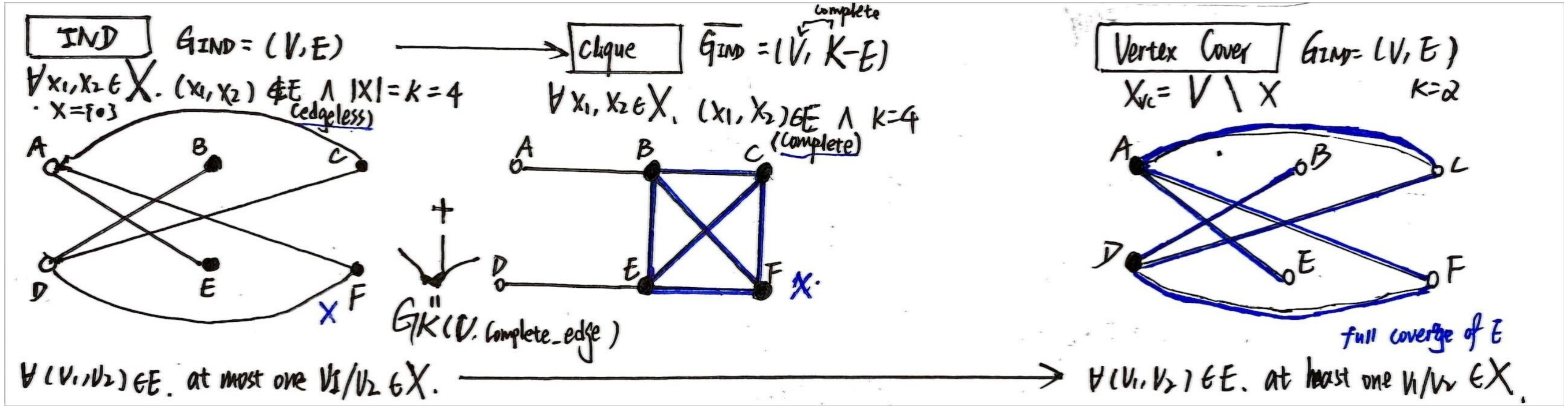
Algorithm	**Input**	**Output**	**Complexity**	**Note**
Evaluation	ϕ, T	$?\mathbb{T}.$	$O(n^2)$	$each\ rule\ O(n)$ remove one variable
SAT	ϕ	$?\exists T.T(X) = \mathbb{T}$	$O(2^n n^2)$	$(\#\ T); \mathcal{NP}$-complete
VAL	ϕ	$?\forall T.T(X) = \mathbb{T}$	$O(2^n n^2)$	$\neg\phi_{S\bar{A}T}$ [negate both IO] $co\text{-}NP$-complete
CVP	DiG	$Circuit\ Value\ \mathbb{B}$	$\mathcal{P}$	linear T by topological sort
CNF-SAT	ϕ_{CNF}	Same as SAT	$\mathcal{NP}$-complete	$SAT <_p CNF - SAT$
3SAT	ϕ_{3CNF}	Same as SAT	$\mathcal{NP}$-complete	$C_{CNF-SAT} = (l_1 \vee l2.. \vee l_k)$ $= (l_1 \vee l2 \vee n) \wedge (\neg n \vee l_3.. \vee l_k)$

Graph Theory

$G : (V, E)$, Directed Acyclic Graph $DiAG$, Undirected Graph UnG,

Node / Vertex $v \in V$, the number of nodes $n = |V|$

Algorithm	Input	Output	Complexity	Note
Reachability	DiG, v_1, v_2	$?\exists p.path(v_1 \to v_2).$	$O(n^2); S(n)$ $\mathcal{NL}$-complete	marked V, neighbours
HAMiltonian	G	$?\exists cycle.path(v_1 \to all!v_i \to v_1)$	$O(n!)$ $\mathcal{NP}$-complete	$3SAT <_p HAM$
TSP	$G, C : V \times V \to N$	order/enum for V HAM with min Cost	$O(n!)/O(2^n n^2)$ $\mathcal{NP}$-complete	$\Omega(n \log n)$ $HAM <_p TSP$
Isomorphism	G_1, G_2	$?\exists f.(v_1, v_2) \in E_1 \Leftrightarrow$ $f(v_1), f(v_2) \in E_2$	$O(n!)$	all possible bijections
k-colourability	G	assignment of colours	$k = 2, \mathcal{P}$ $\mathcal{NP}$-complete	$3SAT <_p 3color$
INDependent Set	$UnG, k = \|X\|$	$?\exists X \subseteq V.\forall x_i.(x_1, x_2) \notin E$ $or\ \forall(v_1, v_2) \in E.$ $v_1, v_2\ at\ most\ one \in X.$	$\mathcal{NP}$-complete	$3SAT <_p IND$
Clique	$UnG, k = \|X\|$	$?\exists X \subseteq V.\forall x_i.(x_1, x_2) \in E$	$\mathcal{NP}$-complete	$\bar{G}_{IND}, X_{IND}$
Vertex Cover	$UnG, k = \|X\|$	$?\exists X \subseteq V.\forall(v_1, v_2) \in E.$ $v_1 \in X \vee v_2 \in X(at\ least\ 1)$	$\mathcal{NP}$-complete	$G_{IND}, V - X_{IND}$



Set Theory

Algorithm	Input	Output	Complexity	Note
Bipartite	$B, G, M \subseteq B \times G$	$?\exists M'.\forall b \in B, g \in G.(b,g) \in M'$ and pairwise disjoint	$\mathcal{P}$	
3D Matching	X, Y, Z, M	similar as above	$\mathcal{NP}$-complete	$3SAT <_p 3DM$
eXact Cover by 3-Sets	$U(3n), S(3) \subset \mathbb{P}(U)$	$?\exists S^*$ pairwise disjoint and full coverage	$\mathcal{NP}$-complete	$U =_{3DM} X \cup Y \cup Z$
Set Cover	$U, S \subset \mathbb{P}(U), n$	$?\exists S^*$ full coverage	$\mathcal{NP}$-complete	$(U_{X3C}, S_{X3C}, \frac{\lvert U_{X3C}\rvert}{3})$ $E(G_{VC}), E(v_i) \lvert deg(v) > 0$